Journal of Statistical Physics, Vol. 85, Nos. 1/2, 1996

Poissonian Asymptotics of a Randomly Perturbed Dynamical System: Flip-Flop of the Stochastic Disk Dynamo

H. M. Ito[1] and T. Mikami[2]

Received October 17, 1995; final January 19, 1996

A dynamical system with two stable equilibrium points will show a flip-flop motion between the neighborhoods of the two points when it is perturbed by small random noises. A typical example is the stochastic disk dynamo model where the two equilibrium points correspond to the two polarities of the earth's magnetic field. We prove what has been suggested by a computer simulation, that the counting process of the flips or the reversals of the earth's field converges to a standard Poisson process if the time is suitably scaled.

KEY WORDS: Poisson; asymptotics; small random perturbation; reversal of the earth's magnetic field.

1. INTRODUCTION

Geophysical phenomena sometimes offer novel stochastic features which are difficult to find in laboratory experiments of limited time span. Analysis of paleomagnetic data reveals that the earth's magnetic field has reversed its polarity many times.[1] One of the most interesting statistical properties of the reversals is that their counting process is a Poisson process.

A random walk model between the two states will be sufficient just to account for the Poisson property. However, from a physical point of view, we need to take account of the dynamo action. The first step will be to find the simplest model which satisfies a minimal requirement for the statistical

[1] Seismology Department, Meteorological Research Institute, Tsukuba, Ibaraki 305, Japan.
[2] Department of Mathematics, Hokkaido University, Sapporo 060, Japan.

0022-4715/96/1000-0041$09.50/0

properties, that is, the symmetric intensity distribution around the two polarities as well as the above Poisson property, together with ergodicity.

In one approach, the statistical properties are attributed to chaos of deterministic systems composed of a couple of disk dynamos. See ref. 2 for present status of this approach. The other approach regards the reversals as a result of perturbations on a single disk dynamo.[3,4] Honkura and Matsushima,[4] in particular, emphasized the importance of the randomness of the perturbations. One of the authors proposed independently a stochastic model,[5] following the recipe by van Kampen[6] and Kubo *et al.*[7] This model, called the stochastic disk dynamo model, describes a system of mutually interacting single disk dynamos, and is expressed by a two-dimensional dynamical system subject to small random perturbations. The unperturbed nongradient dynamical system has two stable equilibrium points corresponding to the two polarities of the earth's magnetic field. The perturbations showing the effects of the eliminated degrees of freedom other than the dipole field are of order of the inverse of the number of interacting disks.

The stochastic disk dynamo model was shown to exhibit a symmetric intensity distribution and ergodicity. Computer simulation suggested the Poisson property, but sometimes resulted in a substantial deviation depending on the system parameters. So it remains unclear whether the stochastic disk dynamo model can serve as a model of reversal fulfilling the minimal requirement mentioned above.

Later, a mathematical justification for the Poisson property was given for a one-dimensional simplified version with a double well potential.[8] The purpose of the present paper is to give a justification to the original two-dimensional model.

Efforts from practical[9,10] as well as mathematical[11,24] viewpoints have been devoted to the asymptotics of dynamical systems subject to small random perturbations. The problem discussed here will be new in two respects. First, instead of a relaxation from a metastable state to a stable state, we are interested in a flip-flop motion between equally stable two states. Second, unlike most work which is concerned with the ensemble averaged properties, we deal with pathwise properties, the importance of which was emphasized by Cassandro *et al.*[12]

We arrange the paper as follows. In Section 2 we give a definition of the stochastic disk dynamo model together with its physical background and mathematical properties. The main result is given in Section 3, Theorem 3.1, which assures that the counting process of the reversals, Y^{ε} of (3.3), converges in the limit of small perturbation to a standard Poisson process if the time is scaled suitably. It is proved by preparing two theorems in Section 3, proofs of which are given in the following subsections.

2. STOCHASTIC DISK DYNAMO MODEL

The stochastic disk dynamo model is described by the following stochastic differential equation for $t \geqslant 0$, $x = (x_1, x_2) \in \mathbf{R}^2$, $\varepsilon > 0$,

$$\begin{aligned} dX^{\varepsilon}(t, x) &= b(X^{\varepsilon}(t, x))\, dt + \varepsilon^{1/2}\, dW(t) \\ X^{\varepsilon}(0, x) &= x \end{aligned} \tag{2.1}$$

where $W(\cdot)$ is a two-dimensional Wiener process,[13] and $b(x) = (b_1(x), b_2(x))$ is given by

$$\begin{aligned} b_1(x_1, x_2) &= -\mu x_1 + x_1 x_2 \\ b_2(x_1, x_2) &= -\nu x_2 + 1 - x_1^2 \end{aligned} \tag{2.2}$$

Here μ and ν are positive constants such that $\nu\mu < 1$.

This model describes a system of several mutually interacting disks. The variables x_1 and x_2 physically stand for the current running through the coil and the angular velocity of the disks, respectively, and $1/\varepsilon$ has a meaning of the number of disks. See ref. 5 for further physical background and some related mathematical arguments.

The deterministic equation (2.1) with $\varepsilon = 0$ is called the single-disk dynamo model, which does not show a reversal of polarity, i.e., a change of the sign of X_1^0. In fact, since $\nu\mu < 1$, $\{X^0(t, x)\}$ has three equilibrium points[14,15]: a hyperbolic equilibrium point

$$H(0, 1/\nu) \tag{2.3}$$

and exponentially stable equilibrium points

$$F_+(\lambda, \mu), \qquad F_-(-\lambda, \mu) \tag{2.4}$$

where

$$\lambda = (1 - \mu\nu)^{1/2} \tag{2.5}$$

On the other hand, the stochastic disk dynamo exhibits reversals. More precisely, the solution to (2.1) exists, is positively recurrent, and has a unique invariant probability density function,[15,16] since

$$\begin{aligned} \langle x, b(x) \rangle &= x_1(-\mu x_1 + x_1 x_2) + x_2(-\nu x_2 + 1 - x_1^2) \\ &= -\mu x_1^2 - \nu x_2^2 + x_2 \\ &\leqslant -\mu x_1^2 - (\nu/2) x_2^2 + 1/(2\nu) \\ &\leqslant -\min(\mu, \nu/2)\, |x|^2 + 1/(2\nu) \end{aligned} \tag{2.6}$$

Here we used

$$x_2 = \nu^{-1/2}\nu^{1/2}x_2 \leqslant \{1/\nu + \nu x_2^2\}/2$$

The regions $\{x \in \mathbf{R}^2: x_1 > \lambda - \delta\}$ and $\{x \in \mathbf{R}^2: x_1 < -\lambda + \delta\}$ are understood as the normal polarity and the reversed polarity of the earth's field, respectively, where $\delta \in (0, \lambda)$ is a parameter to be determined by paleomagnetic observation. We will discuss the asymptotics of the flip-flop motion of X^ε between the two polarities in the limit of $\varepsilon \to 0$.

3. MATHEMATICAL RESULTS

For the stochastic process $\{X^\varepsilon(t, x)\}_{0 \leqslant t < \infty}$ determined by (2.1), we consider a sequence of stopping times $\{\tau_n^\varepsilon(x; \delta)\}_{n \geqslant 0}$ defined by the following: for $\delta \in (0, \lambda)$ and $x = (x_1, x_2) \in \mathbf{R}^2$ for which $|x_1| \geqslant \lambda - \delta$, put

$$\tau_0^\varepsilon(x; \delta) = 0$$

$$\tau_{n+1}^\varepsilon(x; \delta) = \begin{cases} \inf\{t > \tau_n^\varepsilon(x; \delta);\ \lambda - \delta \leqslant X_1^\varepsilon(t, x)\} \\ \qquad \text{if} \quad X_1^\varepsilon(\tau_n^\varepsilon(x; \delta), x) \leqslant -\lambda + \delta \\ \inf\{t > \tau_n^\varepsilon(x; \delta);\ X_1^\varepsilon(t, x) \leqslant -\lambda + \delta\} \\ \qquad \text{if} \quad \lambda - \delta \leqslant X_1^\varepsilon(\tau_n^\varepsilon(x; \delta), x) \end{cases} \tag{3.1}$$

for $n \geqslant 0$.

Take $\beta^\varepsilon(\delta) > 0$ so that

$$P(\tau_1^\varepsilon(F_+; \delta) > \beta^\varepsilon(\delta)) = P(\tau_1^\varepsilon(F_-; \delta) > \beta^\varepsilon(\delta)) = e^{-1} \tag{3.2}$$

which is possible since $(X_1^\varepsilon(\cdot, (x_1, x_2)), X_2^\varepsilon(\cdot, (x_1, x_2)))$ has the same probability law as that of $(-X_1^\varepsilon(\cdot, (-x_1, x_2)), X_2^\varepsilon(\cdot, (-x_1, x_2)))$ [see (2.2)]. Here $F_\pm$ and λ are defined in (2.4) and (2.5), respectively.

Put $\beta^\varepsilon = \beta^\varepsilon(\lambda/2) > 0$, and consider the following stochastic process:

$$Y^\varepsilon(t; x, \delta) = n \qquad \text{if} \quad \tau_n^\varepsilon(x; \delta) \leqslant \beta^\varepsilon t < \tau_{n+1}^\varepsilon(x; \delta) \tag{3.3}$$

Denote by $D_{\mathbf{N}}[0, \infty)$ the space of right continuous functions f: $[0, \infty) \mapsto \mathbf{N}$ with left limits. $D_{\mathbf{N}}[0, \infty)$ is given the Skorohod topology.[21]

The following is our main result.

Theorem 3.1. For any $\delta \in (0, \lambda)$ and any x for which $|x_1| \geqslant \lambda - \delta$, $\{Y^\varepsilon(t; x, \delta)\}_{0 \leqslant t < \infty}$ converges, as $\varepsilon \to 0$, in distribution in $D_{\mathbf{N}}[0, \infty)$ (ref. 21) to a Poisson process with parameter 1.[17]

Theorem 3.1 can be proved by the following Theorems 3.2 and 3.3, proof of which will be given in Sections 3.2 and 3.3. It is standard to prove Theorems 3.2 and 3.3 from Theorem 3.4 in Section 3.1. Theorem 3.2 ensures that any finite dimensional marginal distribution of $\{Y^\varepsilon(t; x, \delta)\}_{0 \leqslant t < \infty}$ converges, as $\varepsilon \to 0$, to that of a Poisson process with parameter 1. Theorem 3.3 and Corollary 7.4, p. 129, of ref. 21 guarantee the tightness of $\{Y^\varepsilon(t; x, \delta)\}_{0 \leqslant t < \infty}$.

Theorem 3.2. For any $\delta \in (0, \lambda)$ and $r > 0$, the following holds: for any $l \in \mathbf{N}$ and any $0 = n_0 \leqslant n_1 \leqslant \cdots \leqslant n_l$,

$$\lim_{\varepsilon \to 0} P(Y^\varepsilon(t_1; x, \delta) = n_1, \ldots, Y^\varepsilon(t_l; x, \delta) = n_l)$$

$$= \prod_{i=1}^{l} (t_i - t_{i-1})^{n_i - n_{i-1}} \exp[-(t_i - t_{i-1})]/(n_i - n_{i-1})! \qquad (3.4)$$

uniformly in $x \in U_r(o) \cap \{y; |y_1| \geqslant \lambda - \delta\}$ and $0 = t_0 < t_1 < \cdots < t_l < \infty$.

Here and in the following, $U_r(x) \equiv \{y \in \mathbf{R}^2; |x - y| < r\}$ for $x \in \mathbf{R}^2$ and $r > 0$, and the symbol o stands for the origin.

Remark 3.1. Roughly speaking, Theorem 3.2 means that $[\tau^\varepsilon_{n+1}(x; \delta) - \tau^\varepsilon_n(x; \delta)]/\beta^\varepsilon$ $(n \geqslant 0)$ are, for sufficiently small $\varepsilon > 0$, independent of each other [from the strong Markov property of $X^\varepsilon(t, x)$], and have exponential distributions, from which we can believe that Theorem 3.1 is true.[17]

Before stating Theorem 3.3, we give some notation.

For $r > 0$, $T > 0$, and $f \in D_N[0, \infty)$, put

$$\omega'(f, r, T) \equiv \inf\{\max_i \sup_{s, t \in [t_{i-1}, t_i)} |f(t) - f(s)|\} \qquad (3.5)$$

where the infimum is taken over all partitions $\{t_i\}$ for which $0 = t_0 < t_1 < \cdots < t_{n-1} < T \leqslant t_n$ with $\min_{1 \leqslant i \leqslant n}(t_i - t_{i-1}) > r$ and over all $n \geqslant 1$.

Theorem 3.3. For any $\delta \in (0, \lambda)$ and $r > 0$, the following holds.

(I) For any $\gamma > 0$ and $t > 0$, there exists $N_{\gamma, t} \in \mathbf{N}$ such that

$$\limsup_{\varepsilon \to 0} (\sup\{P(Y^\varepsilon(t; x, \delta) \geqslant N_{\gamma, t}); |x| < r, |x_1| \geqslant \lambda - \delta\}) < \gamma \qquad (3.6)$$

(II) For any $\gamma > 0$ and $T > 0$, there exists $\tilde{r} > 0$ such that

$$\limsup_{\varepsilon \to 0} (\sup\{P(\omega'(Y^\varepsilon(\cdot; x, \delta), \tilde{r}, T) \geqslant \gamma); |x| < r, |x_1| \geqslant \lambda - \delta\}) < \gamma \qquad (3.7)$$

3.1. Special Case of Theorem 3.2.

In this subsection, we prove Theorem 3.4, which can be proved almost in the same way as Theorem 1 of ref. 22 and which is a special case of Theorem 3.2 with $l=1$ and $n_1=0$, dealing with asymptotics of exit probabilities. Closely related problems have been discussed in refs. 18, 19, and 23.

Theorem 3.4. For any $\delta \in (0, \lambda)$ and $r>0$

$$\lim_{\varepsilon \to 0} P(\tau_1^\varepsilon(x; \delta) > \beta^\varepsilon(\lambda/2)t) = e^{-t} \tag{3.8}$$

uniformly in $x \in U_r(o) \cap \{y; |y_1| \geqslant \lambda - \delta\}$ and $t \geqslant 0$.

Let us explain how to arrange the proof of Theorem 1 of ref. 22 to prove Theorem 3.4.

We note that the following four conditions hold for $R > R_0$ if $R_0 > 0$ is sufficiently large.

(R1) $U_\lambda(F_+) \cup U_\lambda(F_-) \subset U_R(o)$.

(R2) We have

$$\langle x, b(x) \rangle < 0$$

for all $x \in \partial U_R(o)$ [see (2.6)].

(R3) For any $x \in \partial U_R(o) \cap \{y; y_1 \leqslant 0\}$,

$$V_R(x) > V_R(H)$$

Here we put

$$V_R(y) = \inf \left\{ \int_0^t |d\phi(s)/ds - b(\phi(s))|^2 \, ds/2; \phi(0) = F_-, \phi(t) = y, \right.$$

$$\left. \{\phi(s)\}_{0 \leqslant s < t} \subset U_R(o) \cap \{y; y_1 < 0\}, t > 0 \right\}$$

(R4) There exists $\delta_0 \in (0, \lambda)$ such that $V_R(x)$ is smooth in $U_{\delta_0}(F_-)$.

Remark 3.2. For any $r>0$, there exists $R(r)>0$ such that

$$V_R(x) = V_{R'}(x)$$

for $x \in U_r(o) \cap \{y; y_1 < 0\}$ and $R, R' \geqslant R(r)$ (see the proof of Proposition 3.1).

It is easy to see that the conditions R1 and R2 are satisfied. As for R3 and R4, see Proposition 3.1 at the end of this subsection and Theorem 2 in Ref. 20.

Proof of Theorem 3.4 (How to arrange the proof of Theorem 1 of ref. 22.). As the set G in ref. 22, we can take the set $U_R(o)$ for $R > R_o$ from (R1)–(R4) above. In the proof of Lemma 3 in ref. (22), one can take $V_R(x)$ instead of $a(x)$ in ref. 22 and can put $K = \{y;\ V_R(y) \leqslant \alpha\}$ for sufficiently small $\alpha > 0$ from (R4). Then by the argument of the proof of Theorem 1 of ref. 22 and the standard method of Freidlin–Wentzell theory,[11] the proof is complete.QED

Finally, we show that the condition (R3) is satisfied.

Proposition 3.1. We have

$$\liminf_{R \to \infty} (\inf\{V_R(x);\ x \in \partial U_R(o) \cap \{y;\ y_1 \leqslant 0\}\})/R^2 > 0 \tag{3.9}$$

Proof. Put $\kappa = \min(\mu, \nu/2)/2$. Take $R_1 > 0$ such that for $x \notin U_{R_1}(o)$,

$$\langle x, b(x)\rangle \langle -\kappa |x|^2 \tag{3.10}$$

which is possible from (2.6).

For $R > R_1$ and $\{\varphi(s)\}_{0 \leqslant s \leqslant t}$ for which $\varphi(0) = F_-$, $\varphi(t) \in \partial U_R(o) \cap \{y;\ y_1 < 0\}$, and $\{\varphi(s)\}_{0 \leqslant s < t} \subset U_R(o) \cap \{y;\ y_1 < 0\}$, put

$$T(\varphi) \equiv \sup\{s < t;\ \varphi(s) \in U_{R_1}(o) \cap \{y;\ y_1 < 0\}\} \tag{3.11}$$

Then

$$\begin{aligned} &\int_0^t |d\varphi(s)/ds - b(\varphi(s))|^2\, ds/2 \\ &\qquad \geqslant \kappa(|\varphi(t)|^2 - |\varphi(T(\varphi))|^2)/2 = \kappa(R^2 - R_1^2)/2 \end{aligned} \tag{3.12}$$

which completes the proof. This is true, since for $u \in \mathbf{R}^2$ and $x \notin U_{R_1}(o)$,

$$\begin{aligned} |u - b(x)|^2/2 &= \sup_{x \in R^2} [\langle z, u\rangle - \langle b(x), z\rangle - |z|^2/2] \\ &\geqslant \langle \kappa x, u\rangle - \langle b(x), \kappa x\rangle - |\kappa x|^2/2 \\ &\geqslant \langle \kappa x, u\rangle + |\kappa x|^2/2 \geqslant \kappa \langle x, u\rangle \end{aligned} \tag{3.13}$$

from (3.10), and since

$$
\begin{aligned}
&\int_0^t |d\varphi(s)/ds - b(\varphi(s))|^2\, ds/2 \\
&\geqslant \int_{T(\varphi)}^t |d\varphi(s)/ds - b(\varphi(s))|^2\, ds/2 \\
&\geqslant \int_{T(\varphi)}^t \kappa\langle \varphi(s), d\varphi(s)/ds\rangle\, ds \quad [\text{from } (3.13)] \\
&= \kappa(|\varphi(t)|^2 - |\varphi(T(\varphi))|^2)/2 \quad \text{QED}
\end{aligned}
$$

3.2. Proof of Theorem 3.2

In this subsection we prove Theorem 3.2 by induction. First we prove Theorem 3.5, which is a special case of Theorem 3.2 with $l = 1$.

Theorem 3.5. For any $\delta \in (0, \lambda)$ and $r > 0$, the following holds: for any $n \in \mathbf{N}$,

$$\lim_{\varepsilon \to 0} P(Y^\varepsilon(t; x, \delta) = n) = t^n \exp(-t)/n! \tag{3.14}$$

uniformly in $x \in U_r(o) \cap \{y; |y_1| \geqslant \lambda - \delta\}$ and $t \geqslant 0$.

We need the following lemma to prove Theorem 3.5, which can be proved by the standard argument of Freidlin–Wentzell theory, using the strong Markov property of $X^\varepsilon(t; x)$, from Proposition 3.1.

Lemma 3.1. There exists $r_0 > 0$ such that for any $\delta \in (0, \lambda)$, $r > 0$ and $n \in \mathbf{N}$,

$$\lim_{\varepsilon \to 0} P(|X^\varepsilon(\tau_n^\varepsilon(x; \delta), x)| \geqslant r_0) = 0 \tag{3.15}$$

uniformly in $x \in U_r(o) \cap \{y; |y_1| \geqslant \lambda - \delta\}$.

Proof of Theorem 3.5. We prove it by induction on n. When $n = 0$, it has been proved in Theorem 3.4. Suppose that the statement of Theorem 3.5 is true for $n \leqslant k$. Put

$$A_{r,\delta} \equiv U_r(o) \cap \{y; |y_1| \geqslant \lambda - \delta\} \tag{3.16}$$

Then

$$P(Y^\varepsilon(t; x, \delta) = k+1) = P(Y^\varepsilon(t; x, \delta) = k+1, X^\varepsilon(\tau^\varepsilon_{k+1}(x; \delta), x) \in A_{r_0, \delta}) + o(1) \quad (3.17)$$

as $\varepsilon \to 0$, uniformly in $x \in A_{r,\delta}$ and $t \geqslant 0$, from Lemma 3.1.

The second probability on the right hand side of (3.17) is estimated as follows:

$$\begin{aligned} &P(Y^\varepsilon(t; x, \delta) = k+1, X^\varepsilon(\tau^\varepsilon_{k+1}(x; \delta), x) \in A_{r_0, \delta}) \\ &= \int_{0 \leqslant s \leqslant t,\, y \in A_{r_0, \delta}} P(\tau^\varepsilon_{k+1}(x; \delta)/\beta^\varepsilon \in ds, X^\varepsilon(\tau^\varepsilon_{k+1}(x; \delta), x) \in dy) \\ &\quad \times P(\beta^\varepsilon(t-s) < \tau^\varepsilon_1(y; \delta)) \\ &= \int_0^t P(\tau^\varepsilon_{k+1}(x; \delta)/\beta^\varepsilon \in ds) \exp[-(t-s)] + o(1) \end{aligned}$$

as $\varepsilon \to 0$, uniformly in $x \in A_{r,\delta}$ and $t \geqslant 0$, from Theorem 3.4 and Lemma 3.1; and

$$\begin{aligned} &\int_0^t P(\tau^\varepsilon_{k+1}(x; \delta)/\beta^\varepsilon \in ds) \exp[-(t-s)] \\ &= P(\tau^\varepsilon_{k+1}(x; \delta)/\beta^\varepsilon \leqslant t) \\ &\quad - \int_0^t P(\tau^\varepsilon_{k+1}(x; \delta)/\beta^\varepsilon \leqslant s) \exp[-(t-s)]\, ds \\ &= P(Y^\varepsilon(t; x, \delta) \geqslant k+1) - \int_0^t P(Y^\varepsilon(s; x, \delta) \geqslant k+1) \exp[-(t-s)]\, ds \\ &= t^{k+1} \exp(-t)/(k+1)! + o(1) \end{aligned}$$

as $\varepsilon \to 0$, uniformly in $x \in A_{r,\delta}$ and $t \geqslant 0$, by the assumption on induction.

Q.E.D.

Next we prove Theorem 3.2 from Theorem 3.5.

Proof of Theorem 3.2. We prove it by induction. When $l = 1$, it is done by Theorem 3.5. Suppose that the statement of Theorem 3.2 is true when $l = k \geqslant 2$. Then we only have to show the following to complete the proof: for $m = n_1(\geqslant 1)$, $n_1 + 1$,

$$\lim_{\varepsilon\to 0} P(m \leqslant Y^\varepsilon(t_1; x, \delta),\ Y^\varepsilon(t_2; x, \delta) = n_2, \ldots,\ Y^\varepsilon(t_{k+1}; x, \delta) = n_{k+1})$$

$$= \left\{ \sum_{j=m}^{n_2} [(t_1)^j \exp(-t_1)/j!](t_2 - t_1)^{n_2 - j} \exp[-(t_2 - t_1)]/(n_2 - j)! \right\}$$

$$\times \prod_{i=3}^{l} (t_i - t_{i-1})^{n_i - n_{i-1}} \exp[-(t_i - t_{i-1})]/(n_i - n_{i-1})! \tag{3.18}$$

uniformly in $x \in U_r(o) \cap \{y; |y_1| \geqslant \lambda - \delta\}$ and $0 = t_0 < t_1 < \cdots < t_l < \infty$.

Let us prove (3.18). For $m \geqslant 1$,

$$P(m \leqslant Y^\varepsilon(t_1; x, \delta),\ Y^\varepsilon(t_2; x, \delta) = n_2, \ldots,\ Y^\varepsilon(t_{k+1}; x, \delta) = n_{k+1})$$

$$= P(m \leqslant Y^\varepsilon(t_1; x, \delta),\ Y^\varepsilon(t_2; x, \delta) = n_2, \ldots,\ Y^\varepsilon(t_{k+1}; x, \delta) = n_{k+1},$$

$$X^\varepsilon(\tau^\varepsilon_m(x; \delta), x) \in A_{r_0, \delta}) + o(1) \tag{3.19}$$

as $\varepsilon \to 0$, uniformly in $x \in A_{r,\delta}$ and $0 < t_1 < \cdots < t_l < \infty$ from Lemma 3.1.

The probability on the right hand side of (3.19) is estimated as follows:

$$P(m \leqslant Y^\varepsilon(t_1; x, \delta),\ Y^\varepsilon(t_2; x, \delta) = n_2, \ldots,\ Y^\varepsilon(t_{k+1}; x, \delta) = n_{k+1},$$

$$X^\varepsilon(\tau^\varepsilon_m(x; \delta), x) \in A_{r_0, \delta})$$

$$= \int_{0 \leqslant s \leqslant t_1,\ y \in A_{r_0, \delta}} P(\tau^\varepsilon_m(x; \delta)/\beta^\varepsilon \in ds,\ X^\varepsilon(\tau^\varepsilon_m(x; \delta), x) \in dy)$$

$$\times P(Y^\varepsilon(t_2 - s; y, \delta) = n_2 - m, \ldots,\ Y^\varepsilon(t_{k+1} - s; y, \delta) = n_{k+1} - m)$$

[by the strong Markov property of $X^\varepsilon(t, x)$]

$$= \int_0^{t_1} P(\tau^\varepsilon_m(x; \delta)/\beta^\varepsilon \in ds)(t_2 - s)^{n_2 - m} \exp[-(t_2 - s)]/(n_2 - m)!$$

$$\times \prod_{i=3}^{k+1} (t_i - t_{i-1})^{n_i - n_{i-1}} \exp[-(t_i - t_{i-1})]/(n_i - n_{i-1})! + o(1) \tag{3.20}$$

as $\varepsilon \to 0$, uniformly in $x \in A_{r,\delta}$ and $0 < t_1 < \cdots < t_l < \infty$, again by Lemma 3.1 and by the assumption on induction.

From (3.20) we only have to prove that

$$\lim_{\varepsilon \to 0} \int_0^{t_1} P(\tau^\varepsilon_m(x; \delta)/\beta^\varepsilon \in ds)(t_2 - s)^{n_2 - m} \exp[-(t_2 - s)]/(n_2 - m)!$$

$$= \sum_{j=m}^{n_2} [(t_1)^j \exp(-t_1)/j!](t_2 - t_1)^{n_2 - j} \exp[-(t_2 - t_1)]/(n_2 - j)! \tag{3.21}$$

uniformly in $x \in A_{r,\delta}$ and $0 < t_1 < t_2 < \infty$, which is done as

$$\int_0^{t_1} P(\tau_m^\varepsilon(x;\delta)/\beta^\varepsilon \in ds)(t_2-s)^{n_2-m} \exp[-(t_2-s)]/(n_2-m)!$$

$$= P(\tau_m^\varepsilon(x;\delta)/\beta^\varepsilon \leqslant t_1)(t_2-t_1)^{n_2-m} \exp[-(t_2-t_1)]/(n_2-m)!$$

$$-\int_0^{t_1} P(\tau_m^\varepsilon(x;\delta)/\beta^\varepsilon \leqslant s)$$

$$\times \{d[(t_2-s)^{n_2-m} \exp[-(t_2-s)]/(n_2-m)!]/ds\}\, ds$$

$$= \exp(-t_2) \sum_{j=m}^{n_2} [(t_1)^j/j!](t_2-t_1)^{n_2-j}/(n_2-j)! + o(1) \tag{3.22}$$

as $\varepsilon \to 0$, uniformly in $x \in A_{r,\delta}$ and $0 < t_1 < t_2 < \infty$ by Theorem 3.5. QED

3.3. Proof of Theorem 3.3

In this subsection we prove Theorem 3.3.

Proof of (I). For $\gamma > 0$ and $t > 0$ take $N_{\gamma,t} \in \mathbf{N}$ sufficiently large so that

$$\sum_{j=N_{\gamma,t}}^{\infty} t^j \exp(-t)/j! \leqslant \gamma/4 \tag{3.23}$$

Then from Theorem 3.5,

$$P(Y^\varepsilon(t;x,\delta) \geqslant N_{\gamma,t})$$

$$= 1 - P(Y^\varepsilon(t;x,\delta) \leqslant N_{\gamma,t} - 1)$$

$$\leqslant 2 \sum_{j=N_{\gamma,t}}^{\infty} t^j \exp(-t)/t! < \gamma/2$$

for sufficiently small $\varepsilon > 0$, uniformly in $x \in A_{r,\delta}$. QED

Next we prove (II).

Proof of (II). (See ref. 21, p. 134, Lemma 8.2.) For $\gamma > 0$ and $T > 0$, take $N_{\gamma,t}$ which satisfies (3.23) with $t = T$, and take $\tilde{r} \in (0, \gamma/(4N_{\gamma,T}))$. Then

$$
\begin{aligned}
&P(\omega'(Y^\varepsilon(\cdot; x, \delta), \tilde{r}, T) \geqslant \gamma) \\
&\quad \leqslant P(\min_{0 \leqslant k \leqslant Y^\varepsilon(T; x, \delta)} (\tau^\varepsilon_{k+1}(x; \delta) - \tau^\varepsilon_k(x; \delta)) < \beta^\varepsilon \tilde{r}) \\
&\quad \leqslant P(Y^\varepsilon(T; x, \delta) \geqslant N_{\gamma, T}) \\
&\qquad + P(Y^\varepsilon(T; x, \delta) < N_{\gamma, T}, \min_{0 \leqslant k \leqslant Y^\varepsilon(T; x, \delta)} (\tau^\varepsilon_{k+1}(x; \delta) - \tau^\varepsilon_k(x; \delta)) < \beta^\varepsilon \tilde{r}) \\
&\quad \leqslant P(Y^\varepsilon(T; x, \delta) \geqslant N_{\gamma, T}) \\
&\qquad + N_{\gamma, T} \max_{0 \leqslant k \leqslant N_{\gamma, T} - 1} P(\tau^\varepsilon_k(x; \delta) \leqslant \beta^\varepsilon T, \tau^\varepsilon_{k+1}(x; \delta) - \tau^\varepsilon_k(x; \delta) < \beta^\varepsilon \tilde{r})
\end{aligned} \tag{3.24}
$$

The first probability in the last part of (3.24) is less than $\gamma/2$, for sufficiently small $\varepsilon > 0$, uniformly in $x \in A_{r, \delta}$, in the same way as in the proof of (I).

The second probability in the last part of (3.24) is transformed as follows: for $k = 0, \ldots, N_{\gamma, T} - 1$,

$$
\begin{aligned}
&P(\tau^\varepsilon_k(x; \delta) \leqslant \beta^\varepsilon T, \tau^\varepsilon_{k+1}(x; \delta) - \tau^\varepsilon_k(x; \delta) < \beta^\varepsilon \tilde{r}) \\
&= P(\tau^\varepsilon_k(x; \delta) \leqslant \beta^\varepsilon T, \tau^\varepsilon_{k+1}(x; \delta) - \tau^\varepsilon_k(x; \delta) < \beta^\varepsilon \tilde{r}, X^\varepsilon(\tau^\varepsilon_k(x; \delta), x) \in A_{r_0, \delta}) + o(1)
\end{aligned} \tag{3.25}
$$

as $\varepsilon \to 0$, uniformly in $x \in A_{r, \delta}$ from Lemma 3.1.

The probability on the right hand side of (3.25) is estimated as follows:

$$
\begin{aligned}
&P(\tau^\varepsilon_k(x; \delta) \leqslant \beta^\varepsilon T, \tau^\varepsilon_{k+1}(x; \delta) - \tau^\varepsilon_k(x; \delta) < \beta^\varepsilon \tilde{r}, X^\varepsilon(\tau^\varepsilon_k(x; \delta), x) \in A_{r_0, \delta}) \\
&\quad \leqslant \sup_{y \in A_{r_0, \delta}} P(Y^\varepsilon(\tilde{r}; y, \delta) \geqslant 1) \quad [\text{by strong Markov property of } X^\varepsilon(t, x)] \\
&\quad \leqslant 2[1 - \exp(-\tilde{r})] \leqslant 2\tilde{r} < \gamma/(2N_{\gamma, T})
\end{aligned}
$$

for sufficiently small $\varepsilon > 0$ from Theorem 3.5. Q.E.D.

REFERENCES

1. R. T. Merrill and M. W. McElhinny, *The Earth's Magnetic Field* (Academic Press, 1983).
2. M. Hoshi and M. Kono, Rikitake two-disk dynamo system: statistical properties and growth of instability *J. Geophys. Res.* **93B**:11634–11653 (1988).
3. A. Cox, Geomagnetic reversals, *Science* **163**:237–245 (1969).
4. Y. Honkura and M. Matsushima, Fluctuation of the nondipole magnetic field and its implication for the process of geomagnetic polarity reversal in the Cox model, *J. Geophy. Res.* **93B**:11631–11642 (1988).

5. H. M. Ito, Stochastic disk dynamo as a model of reversals of the earth's magnetic field, *J. Stat. Phys.* **53**:19–39 (1988).
6. N. G. van Kampen, Fluctuations in nonlinear systems, in *Fluctuation Phenomena in Solids*, R.E. Burgess, ed. (Academic Press, New York, 1965), Chapter 5.
7. R. Kubo, K. Matsuo, and K. Kitahara, Fluctuation and relaxation of macrovariables, *J. Stat. Phys.* **9**:51–96 (1973).
8. H. M. Ito, M. Tomisaki, and Y. Ogura, Some asymptotic theorems of one-dimensional generalized diffusion processes and their applications to statistical physics, *Trends Stat. Phys.* **1**:179–195 (1994).
9. Z. Schuss, *Theory and Applications of Stochastic Differential Equations* (Wiley, New York, 1980).
10. V. I. Mel'nikov, The Kramers problem: Fifty years of development, *Phys. Rep.* **209**:1–71 (1991).
11. M. I. Freidlin and A. D. Wentzell, *Random Perturbations of Dynamical Systems* (Springer, Berlin, 1984).
12. M. Cassandro, A. Galves, E. Olivieri, and M. E. Vares, Metastable behavior of stochastic dynamics: A pathwise approach, *J. Stat. Phys.* **35**:603–634 (1984).
13. N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion Processes*, 2nd ed. (North-Holland, Amsterdam, and Kodansha, Tokyo, 1989).
14. P. Hartman, *Ordinary Differential Equations* (Wiley, New York, 1964), Chapter 9.
15. H. M. Ito, Ergodicity of randomly perturbed Lorenz model, *J. Stat. Phys.* **35**:151–158 (1984).
16. R. Z. Has'minskii, *Stochastic Stability of Differential Equations*, (Sijthoff & Noordhoff, 1980), Chapter 4.
17. L. Breiman, *Probability* (SIAM, 1992).
18. M. V. Day, Exponential levelling of stochastically perturbed dynamical systems, *SIAM J. Math. Anal.* **13**:532–540 (1982).
19. M. V. Day, On the exponential exit law in the small parameter exit problem, *Stochastics* **8**:297–323 (1983).
20. M. V. Day and T. A. Darden, Some regularity results on the Ventcel–Freidlin quasipotential functions, *Appl. Math. Optim.* **13**:259–282 (1985).
21. S. N. Ethier and T. G. Kurtz, *Markov Processes: Characterization and Convergence* (Wiley, New York, 1986).
22. A. Galves, E. Olivieri, and M. E. Vares, Metastability for a class of dynamical systems subject to small random perturbations, *Ann. Prob.* **15**:1288–1305 (1987).
23. Y. Kifer, The exit problem for small random perturbations of dynamical systems with a hyperbolic fixed point, *Israel J. Math.* **40**:74–96 (1981).
24. A. D. Wentzell and M. I. Freidlin, On small random perturbations of dynamical systems, *Russ. Math. Survey* **25**:1–55 (1970).